Quantum state engineering via unitary transformations

A. Vidiella-Barranco and J.A. Roversi

Instituto de Física "Gleb Wataghin", Universidade Estadual de Campinas, 13083-970 Campinas SP Brazil

(February 1, 2008)

Abstract

We construct a Hamiltonian for the generation of arbitrary pure states of the quantized electromagnetic field. The proposition is based upon the fact that a unitary transformation for the generation of number states has been already found. The general unitary transformation here obtained, would allow the use of nonlinear interactions for the production of pure states. We discuss the applicability of this method by giving examples of generation of simple superposition states. We also compare our Hamiltonian with the one resulting from the interaction of trapped ions with two laser fields.

42.50.Ct, 42.50.Dv

The generation of pure nonclassical states of the quantized electromagnetic field is a central topic in quantum optics. Several schemes have been already proposed, based either on the *non-unitary* collapse of the state vector (through atom measurements) within micromaser environments [1,2] or by using a cavity QED *unitary* time-dependent interaction [3]. Both approaches involve individual atoms interacting with a single-mode cavity field, what would demand extraordinary control in a generation experiment. It is therefore interesting to seek for alternative methods for the generation of nonclassical light. A significant advance was the proposal, for the first time, of a unitary generator for the number state $|n\rangle$ by Kilin and Horoshko [4]. They derived an expression for a Hamiltonian $\hat{H}_n$ such that $\hat{H}_n|0\rangle = |n\rangle$ and $\hat{H}_n|n\rangle = |0\rangle$, i.e., they obtained a Hamiltonian that generates the state $|n\rangle$ from the vacuum state $|0\rangle$, but only during particular interaction times $[t_m = (m + 1/2)\pi]$. This could be experimentally achieved by pumping conveniently prepared nonlinear media, as discussed in [4].

In this paper we propose a generalization of Kilin-Horoshko's procedure for the generation of an arbitrary pure state $|\Psi\rangle$, which for instance, can be expressed as a linear superposition of number states

$$|\Psi\rangle = \sum_{n=0}^{M} C_n \, |n\rangle, \tag{1}$$

where M is the maximum number of photons in the radiation field. For certain interaction times, we assume the existence of a Hamiltonian $\hat{H}_{|\Psi\rangle}$ such that $\hat{H}_{|\Psi\rangle}|0\rangle = |\Psi\rangle$ and $\hat{H}_{|\Psi\rangle}|\Psi\rangle = |0\rangle$. The Hamiltonian $\hat{H}_{|\Psi\rangle}$ may be written in the following general form

$$\hat{H}_{|\Psi\rangle} = \sum_{n=0}^{M} C_n \left(|0\rangle\langle n| + |n\rangle\langle 0|\right) - C_0 \sum_{k=0}^{M} |k\rangle\langle k| + \hat{P}\hat{F}\hat{P}, \tag{2}$$

where $\hat{P} = \hat{I} - \sum_{k=0}^{M} |k\rangle\langle k|$ and $\hat{F}$ is any Hermitian operator. The sums of projectors $\sum_{k=0}^{M} |k\rangle\langle k|$ are needed in order to compensate all the terms present in the superposition state in Eq.(1). For the sake of simplicity we have considered the coefficients C_n real. Our task now is to find an operator $\hat{F}$ such that we are able to delete the higher order powers of $\hat{a}$ and $\hat{a}^\dagger$ in order to obtain the simplest possible expression for $\hat{H}_{|\Psi\rangle}$. We start by decomposing $\hat{F}$ in a convenient way

$$\hat{F}(\hat{a}, \hat{a}^\dagger) = \hat{f}_0(\hat{a}^\dagger\hat{a}) + \sum_{m=1}^{M} \left[\hat{f}_m(\hat{a}^\dagger\hat{a}) \frac{\hat{a}^m}{\sqrt{m!}} + \frac{\hat{a}^{\dagger m}}{\sqrt{m!}} \hat{f}_m^*(\hat{a}^\dagger\hat{a}) \right]. \tag{3}$$

Now if we substitute Eq.(3) and the expression for $\hat{P}$ into Eq.(2), after some algebra we find the conditions that the functions $f_l(k) \equiv \langle k|\hat{f}_l(\hat{a}^\dagger\hat{a})|k\rangle$ must obey in order to cancel most of the terms in $\hat{H}_{|\Psi\rangle}$. These conditions are

$$f_0(0) = C_0; \qquad f_0(k) = -C_0; \qquad f_m(0) = C_m; \qquad f_m(k) = 0, \tag{4}$$

for $k = 1, 2, \ldots M$ and $m = 1, 2, \ldots M$. Those are necessary but not sufficient conditions for the determination of the functions $\hat{f}_m(\hat{a}^\dagger\hat{a})$, what introduces a certain degree of arbitrarieness in their choice. A simple form for $\hat{f}_0$ and $\hat{f}_m$ involving a *finite expansion* in the annihilation and creation operators $\hat{a}$ and $\hat{a}^\dagger$ is

$$\hat{f}_0(\hat{a}^\dagger\hat{a}) = C_0 \left[2 \left(1 - \hat{a}^\dagger\hat{a} \right) \mathcal{F}(\hat{a}^\dagger\hat{a}) - 1 \right]; \qquad \hat{f}_m(\hat{a}^\dagger\hat{a}) = C_m \, \mathcal{F}(\hat{a}^\dagger\hat{a}); \qquad m \neq 0, \tag{5}$$

where

$$\mathcal{F}(\hat{a}^\dagger\hat{a}) = \sum_{l=0}^{M} A_l \, (\hat{a}^\dagger\hat{a})^l. \tag{6}$$

The condition $f_m(0) = C_m$ demands that $\mathcal{F}(\hat{a}^\dagger\hat{a})|0\rangle = |0\rangle$, which means that we must have $A_0 = 1$. Moreover, we can use the condition $f_m(k) = 0$ [see Eq.(4)] to determine the remaining coefficients A_l. The successive application of the function $\mathcal{F}(\hat{a}^\dagger\hat{a})$ onto the M number states $|1\rangle$, $|2\rangle \ldots |M\rangle$ gives rise to the following M coupled linear equations for the coefficients A_l:

$$\begin{aligned}
&1 + A_1 + A_2 + \ldots + A_M = 0 \\
&1 + 2A_1 + 4A_2 + \ldots + 2^M A_M = 0 \\
&\vdots \\
&1 + MA_1 + M^2 A_2 + \ldots + M^M A_M = 0.
\end{aligned} \tag{7}$$

The solution of this set of equations always exists, what completely determines the functions $\hat{f}_m$. Therefore there is a specific set of numerical coefficients A_l for every value of M, and

which are of the form: $A_1 = \alpha_1/M!, A_2 = \alpha_2/M!, \ldots A_M = \alpha_M/M!$, with $|\alpha_1| < |\alpha_2| < \ldots < |\alpha_{M-1}| < |\alpha_M|$. In particular, $\alpha_M = (-1)^M/M!$. This result is convenient for us because the powers of $\hat{a}^\dagger\hat{a}$ will be multiplied by increasingly smaller coefficients, i.e., the relative importance of the higher-order terms will be consistently diminished.

The complete and already simplified Hamiltonian will then read

$$\hat{H}_{|\Psi\rangle} = f_0\left(\hat{a}^\dagger\hat{a}\right) + \sum_{m=1}^{M} \frac{C_m}{\sqrt{m!}}\left[\mathcal{F}(\hat{a}^\dagger\hat{a})\,\hat{a}^m + \hat{a}^{\dagger m}\,\mathcal{F}(\hat{a}^\dagger\hat{a})\right]. \tag{8}$$

We would like to remark that the function $f_0(\hat{a}^\dagger\hat{a})$ is important for the establishment of the match between our Hamiltonian and the "physical" interaction Hamiltonian in a realistic experiment. Here, we have made a choice for f_0 [see Eq.(5)] that allows the inclusion of the Kerr effect in the generation process, as we are going to show.

In what follows we discuss applications of this method. It would be interesting to generate a state exhibiting nonclassical properties such as squeezing, antibunching and sub-Poissonian character, for instance. These effects occur simultaneously in a single state, namely the binomial state [5], which admits an expansion as the one in Eq.(1), having a finite number of number state (real) coefficients different from zero

$$C_n = B_n^M = \left[\frac{M!}{n!\,(M-n)!}\,p^n(1-p)^{M-n}\right]^{1/2}. \tag{9}$$

Its photon number distribution $P_n = (B_n^M)^2$ is a binomial distribution. The binomial states are caracterized by two parameters: p being the probability of emission of a single photon, and M the maximum number of photons in the field. They interpolate between a number state $|M\rangle$ (containing M photons), as $p \to 1$, and a coherent state $|\alpha\rangle$ (with real amplitude $\alpha = \sqrt{pM}$) as $p \to 0$ and $M \to \infty$, i.e., they belong to a class of "intermediate states". Their nonclassical properties, of course, are strongly dependent on the values of the "interpolation parameters" p and M [6]. Generalizations of binomial states also include the squeezed coherent states [7].

We start by showing an explicit calculation of the Hamiltonian for the particular case of $M = 1$, i.e., the state being generated would be the superposition of the vacuum state with

the one-photon state $|\phi\rangle = C_0|0\rangle + C_1|1\rangle$. In this case we have that $\mathcal{F}(\hat{a}^\dagger\hat{a}) = 1 + A_1\hat{a}^\dagger\hat{a}$. From the requirement $(1 + A_1\hat{a}^\dagger\hat{a})|1\rangle = 0$ [fourth condition in Eq.(4)], we obtain $A_1 = -1$. The Hamiltonian in Eq.(8) then becomes

$$\hat{H}_{|\phi\rangle} = (1-p)^{1/2}\left[1 - 4\hat{a}^\dagger\hat{a} + 2\left(\hat{a}^\dagger\hat{a}\right)^2\right] + p^{1/2}\left[(1-\hat{a}^\dagger\hat{a})\hat{a} + \hat{a}^\dagger(1-\hat{a}^\dagger\hat{a})\right]. \tag{10}$$

On the other hand, if we pump with a classical field $E_c = Ee^{-i\Omega t} + E^* e^{i\Omega t}$ (polarized along the y direction) a nonlinear medium characterized by linear and nonlinear susceptibilities $\chi^{(1)}$ and $\chi^{(3)}$ respectively, the coupling of the signal (polarized along the x direction), the pump and the output fields will be described by the following Hamiltonian [4]:

$$\begin{aligned}\hat{H}_{NL} = {} & \chi^{(1)}_{xx}\hat{a}^\dagger\hat{a} + \chi^{(1)}_{yy}|E|^2 + +\chi^{(3)}_{xxxx}\hat{a}^{\dagger 2}\hat{a}^2 + \chi^{(3)}_{xyxy}\hat{a}^\dagger\hat{a}|E|^2 + \chi^{(3)}_{yyyy}|E|^2|E|^2 \\ & + \left(\chi^{(1)}_{xy}\hat{a}^\dagger E + \chi^{(3)}_{xxyy}\hat{a}^{\dagger 2}E^2 + \chi^{(3)}_{xxxy}\hat{a}^{\dagger 2}\hat{a}E + \chi^{(3)}_{xyyy}\hat{a}^\dagger|E|^2E + \text{H.c.}\right).\end{aligned} \tag{11}$$

Both the pump and signal are travelling waves propagating along the z axis. The expression for our generator [in Eq.(10)] has the same form as the Hamiltonian in Eq.(11), what allows, at least in principle, to know how to choose and prepare a nonlinear medium in such a way that we end up with the Hamiltonian in Eq.(10), as we are going to show. The linear and nonlinear susceptibilities have to assume specific values in order to make possible the correspondence between both Hamiltonians. For instance, there is no term in Eq.(10) proportional to $\hat{a}^{\dagger 2}$, which means that we must have $\chi^{(3)}_{xxyy} = 0$. By comparing terms proportional to $\hat{a}^\dagger$, $\hat{a}^{\dagger 2}\hat{a}$ and to $\hat{a}^\dagger\hat{a}$, we obtain

$$\chi^{(1)}_{xy}(E_0)E + \chi^{(3)}_{xyyy}|E|^2E = -\chi^{(3)}_{xxxy}E = p^{1/2}, \tag{12}$$

and

$$\chi^{(1)}_{xx}(E_0) + \chi^{(3)}_{xyxy}|E|^2 = -\chi^{(3)}_{xxxx} = -2(1-p)^{1/2}. \tag{13}$$

These relations among the different susceptibilities are of the same form as the ones found in reference [4]. However, in our case we are generating a state which is the one-photon state coherently superposed to the vacuum, fact which is embodied in the "tuning" parameter

p. The susceptibilities $\chi^{(3)}_{ijkl}$ are of course a feature of the chosen crystal and the first-order susceptibilities $\chi^{(1)}_{ij}$ can be changed by the application of a static field E_0. We can control the generation procedure by adjusting the values of the pump field E and the static field E_0 in order to satisfy the relations (12) and (13). From Eq.(12) we see that the probability of having one photon in the field, p, is proportional to the pump field amplitude E, as one would expect. We note that the conditions in Eq.(12) and (13) connect the quantum superposition principle, represented by the coherent superposition of the vacuum with the one-photon state, with "macroscopic features" such as nonlinear susceptibilities in a crystal.

Our general scheme allows the generation, in principle, of virtually any pure state of the quantized field through some kind of nonlinear interaction. As a second example, we now construct the generating Hamiltonian for the superposition of the vacuum state $|0\rangle$ with the two-photon state $|2\rangle$, or $|\psi\rangle = C_0|0\rangle + C_2|2\rangle$. In this case the solution of the system of equations in Eq.(7) gives us $A_1 = -3/2$, $A_2 = 1/2$, and the Hamiltonian will read

$$\begin{aligned}\hat{H}_{|\psi\rangle} &= C_0\left[1 - 5(\hat{a}^\dagger\hat{a}) + 4(\hat{a}^\dagger\hat{a})^2 - 1(\hat{a}^\dagger\hat{a})^3\right] + \frac{C_2}{\sqrt{2}}\left[\mathcal{F}(\hat{a}^\dagger\hat{a})\hat{a}^2 + \hat{a}^{\dagger 2}\mathcal{F}(\hat{a}^\dagger\hat{a})\right];\\ \mathcal{F}(\hat{a}^\dagger\hat{a}) &= 1 - \frac{3}{2}\hat{a}^\dagger\hat{a} + \frac{1}{2}(\hat{a}^\dagger\hat{a})^2. \end{aligned} \tag{14}$$

We note that because the target state $|\psi\rangle$ above does contain even photon numbers only, there are fewer terms in the generating Hamiltonian, what means that additional conditions must be imposed on the relevant susceptibilities in the corresponding nonlinear Hamiltonian. More specifically, in Eq.(14) we have terms of the type $\hat{a}^{\dagger 2}\left(\hat{a}^\dagger\hat{a}\right)^2$, which means that a fourth-order nonlinear susceptibility ($\chi^{(4)}$) should take part in the generation process. Recently there have been developments both theoretical [8] and experimental [9] in processes involving five-wave mixing in fluids. This kind of medium has several advantages, such as a flexible geometry, for instance. However, due to its intrinsic isotropy, processes involving modes in co-linear propagation are forbidden. In general there are more stringent conditions over fluid media than in crystals. This of course may favour our scheme in the sense that we have only a few terms present in our Hamiltonian.

We would expect that the Hamiltonian in Eq.(8) for the generation of binomial states (in

the limits of $p \to 0$ and $M \to \infty$) should be somehow equivalent to Glauber's displacement operator $\hat{D}(\alpha) = \exp(\alpha\hat{a}^\dagger - \alpha^*\hat{a})$. In fact we have

$$\hat{H}_{|\Psi\rangle\to|\alpha\rangle}|0\rangle = \lim_{p\to 0, M\to\infty}\left[B_0^M|0\rangle + \sum_{m=1}^{\infty} B_m^M \frac{\hat{a}^{\dagger m}}{\sqrt{m!}}|0\rangle\right] = \sum_{m=0}^{\infty}\frac{e^{-\alpha^2/2}\alpha^m\hat{a}^{\dagger m}}{m!}|0\rangle$$

$$= e^{-\alpha^2/2}e^{\alpha\hat{a}^\dagger}|0\rangle = e^{-\alpha^2/2}e^{\alpha\hat{a}^\dagger}e^{\alpha^*\hat{a}}|0\rangle = \hat{D}(\alpha)|0\rangle = |\alpha\rangle, \tag{15}$$

i.e., the application of either the appropriate Hamiltonian $\hat{H}_{|\Psi\rangle}$ or $\hat{D}(\alpha)$ onto the vacuum state leads to the same (coherent) state $|\alpha\rangle$. It is also worth verifying that the Hamiltonian in Eq.(8) (in the limit of $p \to 1$) is equivalent to Kilin-Horoshko's Hamiltonian [4] for the generation on number states. In this case $B_n^M = \delta_{n,M}$, and

$$\hat{H}_{|\Psi\rangle\to|M\rangle} = \frac{1}{\sqrt{M!}}\left[\mathcal{F}(\hat{a}^\dagger\hat{a})\hat{a}^M + \hat{a}^{\dagger M}\mathcal{F}(\hat{a}^\dagger\hat{a})\right], \tag{16}$$

so that

$$\hat{H}_{|M\rangle}|0\rangle = \frac{1}{\sqrt{M!}}\hat{a}^{\dagger M}\left[1 + \sum_{l=1}^{M} A_l(\hat{a}^\dagger\hat{a})^l\right]|0\rangle = |M\rangle. \tag{17}$$

The difference between Kilin-Horoshko's Hamiltonian and ours with respect to the number state case rests on the functions $\mathcal{F}(\hat{a}^\dagger\hat{a})$. They do not need to contain higher order powers of $(\hat{a}^\dagger\hat{a})$. In fact a simpler function $\mathcal{F} = 1 - \hat{a}^\dagger\hat{a}/M$ is enough for the number state generation [4]. However, we shall stress that in a real experiment, a nonlinear medium has in general all its nonlinear susceptibilities excited as it is pumped. This means that it is convenient to have the contributions of all powers of $\hat{a}^\dagger\hat{a}$ (up to the Mth) in the interaction Hamiltonian, as it does happen in our approach. Our generalization, then, is consistent with previously known results.

The effective implementation of this technique would constitute a challenging experimental problem, starting from the design of appropriate media. It would be interesting a comment on an alternative physical system, other than the electromagnetic field, that could be suitable for the accomodation of such a nonlinear Hamiltonian. Recently there has been a great deal of interest in the generation of nonclassical states of motion of a single ion

confined into an electromagnetic trap. There have been carried out successful experiments for the production of number, coherent and squeezed vacuum states of the ion's motion [10]. On the other hand, Raman-type excitation of a trapped ion via two laser fields produces an intrinsically nonlinear Hamiltonian, which is conveniently expressed (one-dimensional case), in the following form [11]:

$$\hat{H}_{ion} = \frac{1}{2}\hbar\Omega \hat{f}_k(\hat{n};\eta)(i\eta\hat{a})^k + H.c.; \qquad \text{where} \qquad \hat{f}_k(\hat{n};\eta) = e^{-\eta^2/2}\sum_{l=0}^{\infty}\frac{(-1)^l\eta^{2l}}{l!(l+k)!}\,\hat{a}^{\dagger l}\hat{a}^l. \tag{18}$$

Here $\eta = 2\pi a_0/\lambda$ is the Lamb-Dicke parameter, being $a_0(\equiv 1/\sqrt{2m\nu_1})$ the size of the ground state of the harmonic potential of natural frequency ν_1, $\Omega = \Omega_1\Omega_2^*/2(\omega_{21}-\omega_L)$ is the system's effective two-photon Rabi frequency, Ω_i are the single-photon Rabi frequencies, ω_{21} the atomic transition frequency, and ω_L is the laser frequency. The detuning between the lasers is $\Delta = k\nu_1$ $(k = 1, 2, \ldots)$. This decomposition in Eq.(18) is similar to the one we used in our Hamiltonian [Eq.(8)], the difference being that in the ion's case the functions $\hat{f}_k(\hat{n})$ are already fully specified. For instance, if we consider $\eta \ll 1$, and $k = 1$, we may write

$$\hat{f}_1(\hat{n};\eta) \approx e^{-\eta^2/2}\left(1 - \frac{\eta^2}{2}\hat{a}^\dagger\hat{a}\right), \tag{19}$$

and the interaction Hamiltonian will read

$$\hat{H}_{ion} = \frac{1}{2}i\hbar\eta e^{-\eta^2/2}\left[\Omega\left(1 - \frac{\eta^2}{2}\hat{a}^\dagger\hat{a}\right)\hat{a} - \Omega^*\hat{a}^\dagger\left(1 - \frac{\eta^2}{2}\hat{a}^\dagger\hat{a}\right)\right]. \tag{20}$$

The Hamiltonian above has the same form as the one in Eq.(10), for the generation of the one-photon state $(p \to 1)$. We note that despite the similarity between both Hamiltonians, they are not totally equivalent, because for that we must have $\eta = \sqrt{2}$, which is in contradiction with our assumption $\eta \ll 1$, apart form other differences. Therefore it is not so evident the correspondence between the quantum state engineering we are here proposing, and highly nonlinear systems such as driven trapped ions. Nevertheless, the approach adopted here, based on fixed interaction times, should be in our opinion further investigated in order to seek for a clearer connection between our tailor-made Hamiltonian and a scheme of excitation of ions by laser beams.

We have presented here the explicit construction of a Hamiltonian $\hat{H}_{|\Psi\rangle}$ which is a generator of nonclassical states of the quantized electromagnetic field. This extends the class of methods being considered for this purpose, drawing together, as particular cases, previously known generators such as the number state as well as the coherent state ones. Having a generation scheme based on unitary transformations in hands, new possibilities arise for the engineering of nonclassical light under feasible controlled circumstances, e.g., through nonlinear interactions. Moreover, this scheme has the advantage of involving travelling waves, what obviously minimizes the destructive effect of dissipation.

ACKNOWLEDGMENTS

We would like to thank M. Marchiolli and one of the referees for valuable comments. This work was partially supported by CNPq (Conselho Nacional para o Desenvolvimento Científico e Tecnológico, Brazil).

REFERENCES

[1] K. Vogel, V.M. Akulin and W.P. Schleich, Phys. Rev. Lett. **71,** 1816 (1993).

[2] B.M. Garraway, B. Sherman, H. Moya-Cessa, P.L. Knight and G. Kurizki, Phys. Rev. A **49,** 535 (1994).

[3] C.K. Law and J.H. Eberly, Phys. Rev. Lett. **76,** 1055 (1996).

[4] S. Kilin and D. Horoshko, Phys. Rev. Lett. **74,** 5206 (1995).

[5] D. Stoler, B.E.A. Saleh and M.C. Teich, Optica Acta **32,** 345 (1985).

[6] A. Vidiella-Barranco and J.A. Roversi, Phys. Rev. A **50,** 5233 (1994).

[7] H-C. Fu and R. Sasaki, J. Phys. A **29,** 5637 (1996).

[8] L.C. Dávila Romero, S.R. Meech, and D.L. Andrews, J. Phys. A **30**, 5609 (1997).

[9] A.V. Balakin, N.I. Koroteev, A. Pakulev, A.P. Shkurinov, D. Boucher, P. Masselin, and E. Fertein, JETP Lett. **64**, 718 (1996).

[10] D.M. Meekhof, C. Monroe, B.E. King, W.M. Itano and D.J. Wineland, Phys. Rev. Lett. **76,** 1796 (1996).

[11] S.Wallentowitz and W.Vogel, Phys. Rev. A **55,** 4438 (1997).